Exploration of Curves

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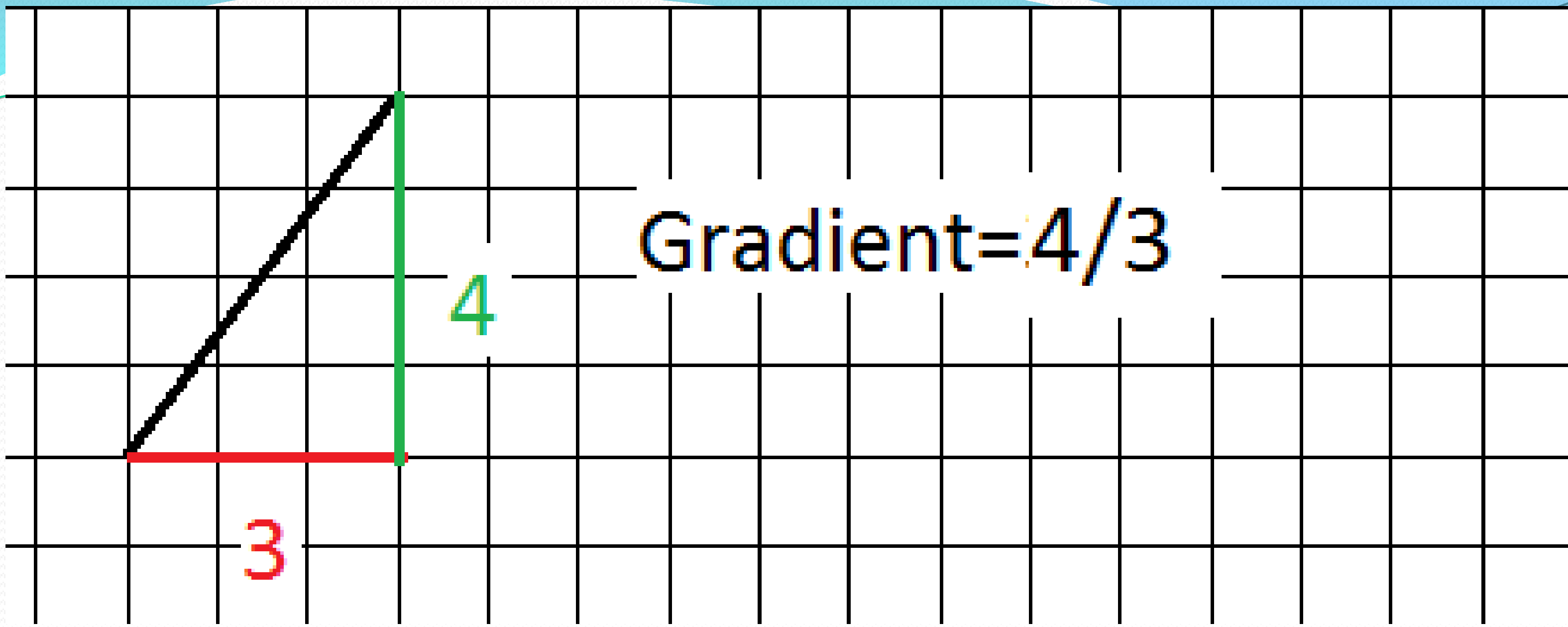
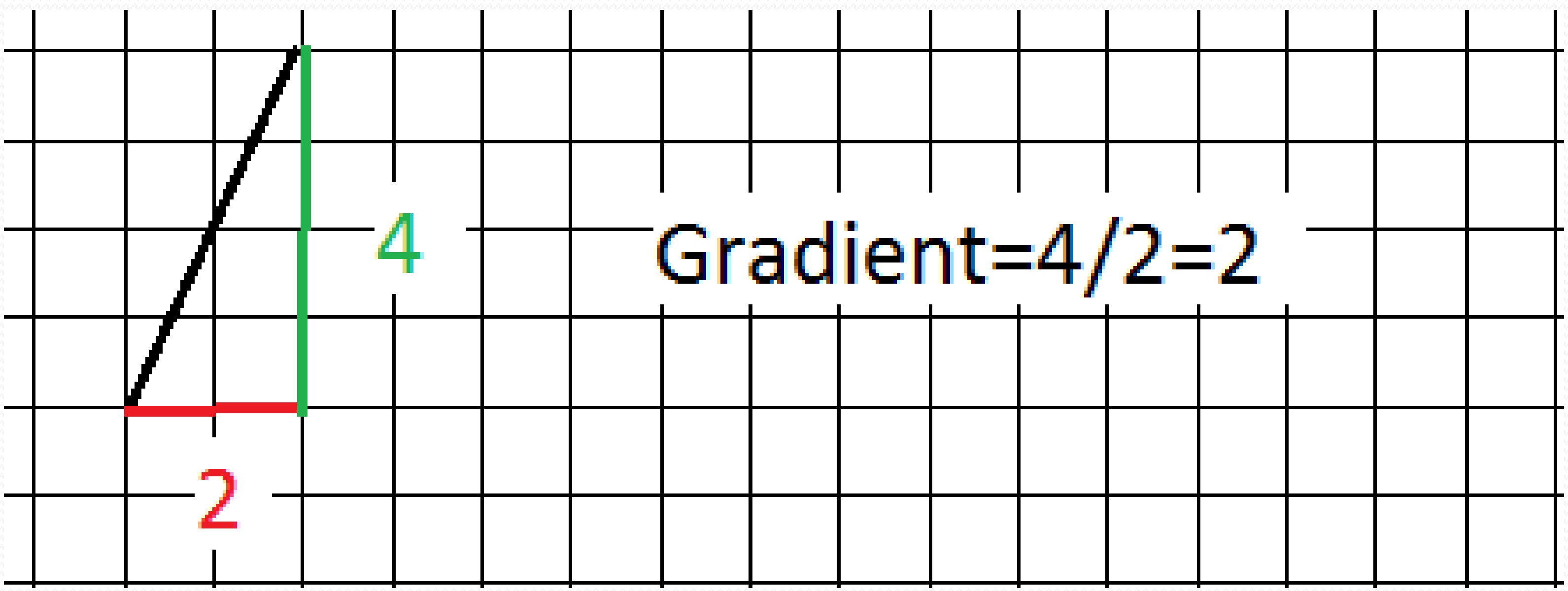
GRADIENT

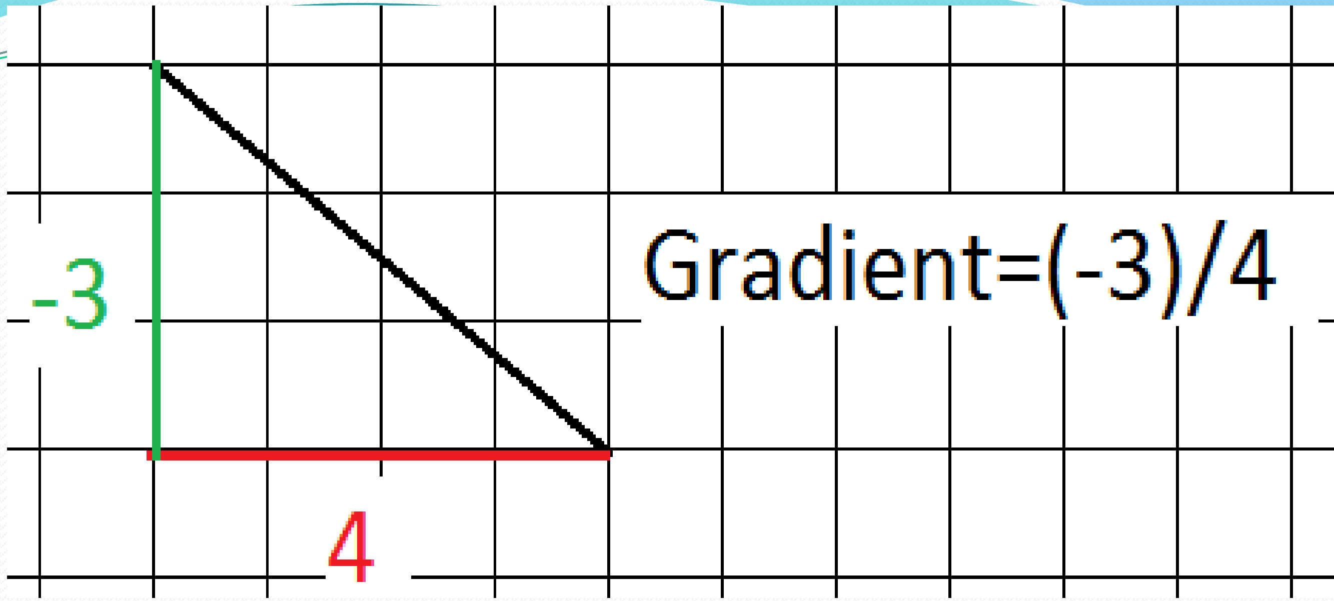
For the topics we are going to look at today we are going to need to know about the gradient.

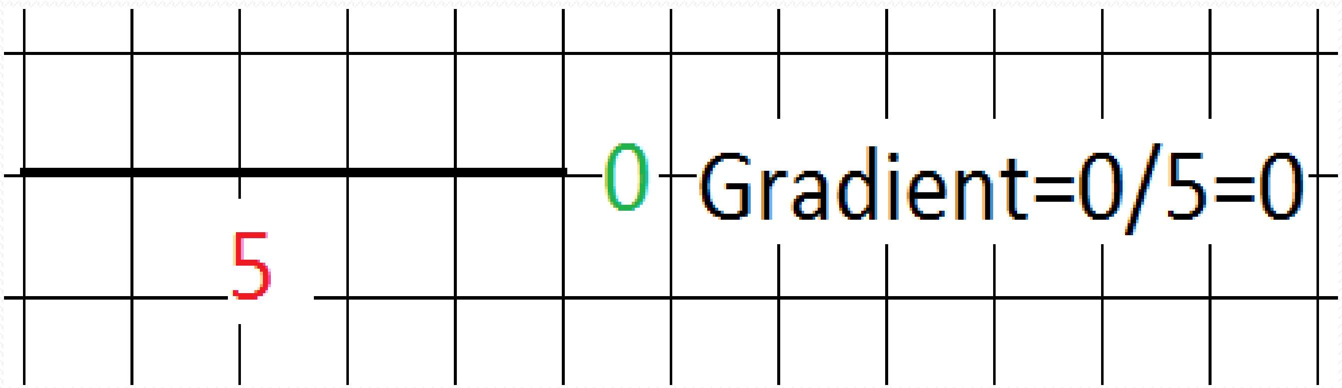
$$gradient = \frac{change\ in\ y}{change\ in\ x}$$

We can look at some examples for straight lines

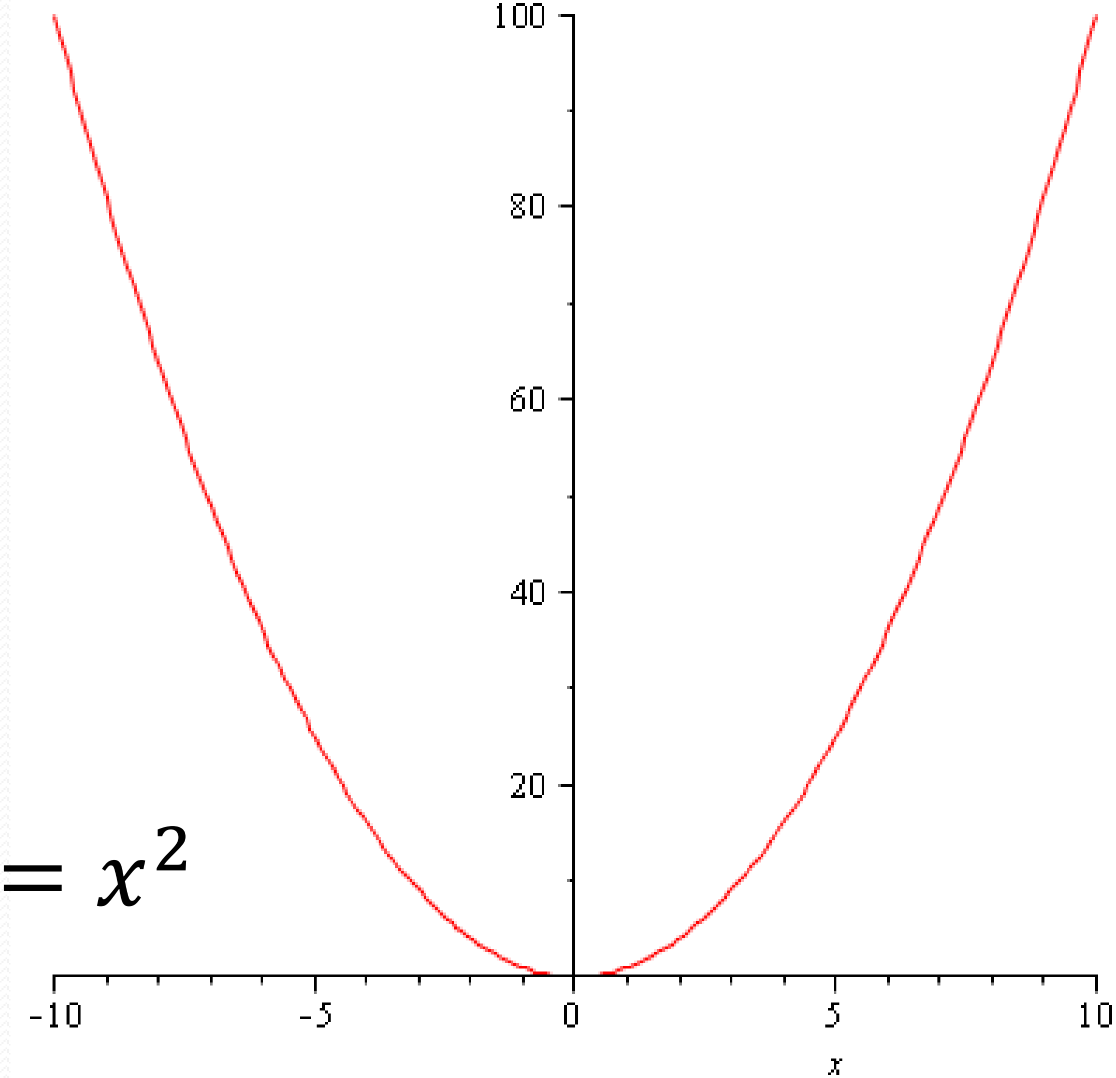

Gradient=4/3
4
3


Gradient=4/2=2
4
2


-3
4
Gradient=(-3)/4

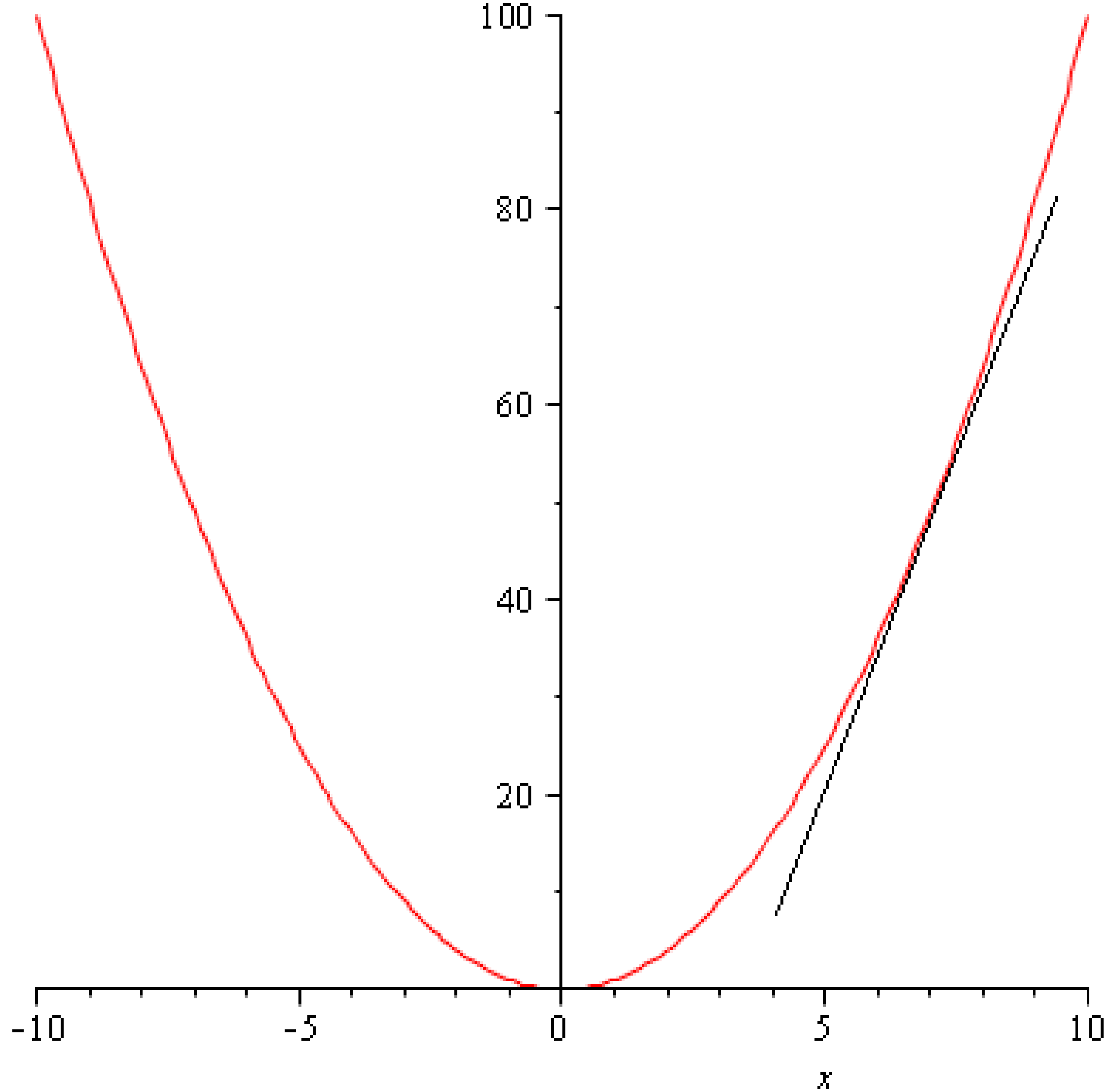

0
5
Gradient=0/5=0

GRADIENT OF CURVES



$$y = x^2$$

GRADIENT OF CURVES



ACTIVITY

GRADIENT FUNCTION

We are now going to define the gradient as:

$$\frac{dy}{dx}$$

This gives us a function which when we input values of x we find the gradient at the point (x, y).

1. Can you spot how we get from one to the other?

2. More advanced, can you work out a rule to link the function, y with its derivative, $\frac{dy}{dx}$?

1.

a) $y = x^2$ $\quad\quad \frac{dy}{dx} = 2x$

b) $y = x^3$ $\quad\quad \frac{dy}{dx} = 3x^2$

c) $y = x^4$ $\quad\quad \frac{dy}{dx} = 4x^3$

d) $y = 2x^4$ $\quad\quad \frac{dy}{dx} = 8x^3$

e) $y = 3x^3$ $\quad\quad \frac{dy}{dx} = 9x^2$

f) $y = 2x^2 + 7x^3$ $\quad\quad \frac{dy}{dx} = 4x + 21x^2$

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2. What is $\frac{dy}{dx}$ when $y = ax^n$?

THE FORMULA

$$y = ax^n$$

$$\frac{dy}{dx} = nax^{n-1}$$

(a is a constant)

3. Using the rule calculate $\frac{dy}{dx}$ for $y = x$

$$\frac{dy}{dx} = nax^{n-1}$$

4. What is $\frac{dy}{dx}$ when $x = constant$?

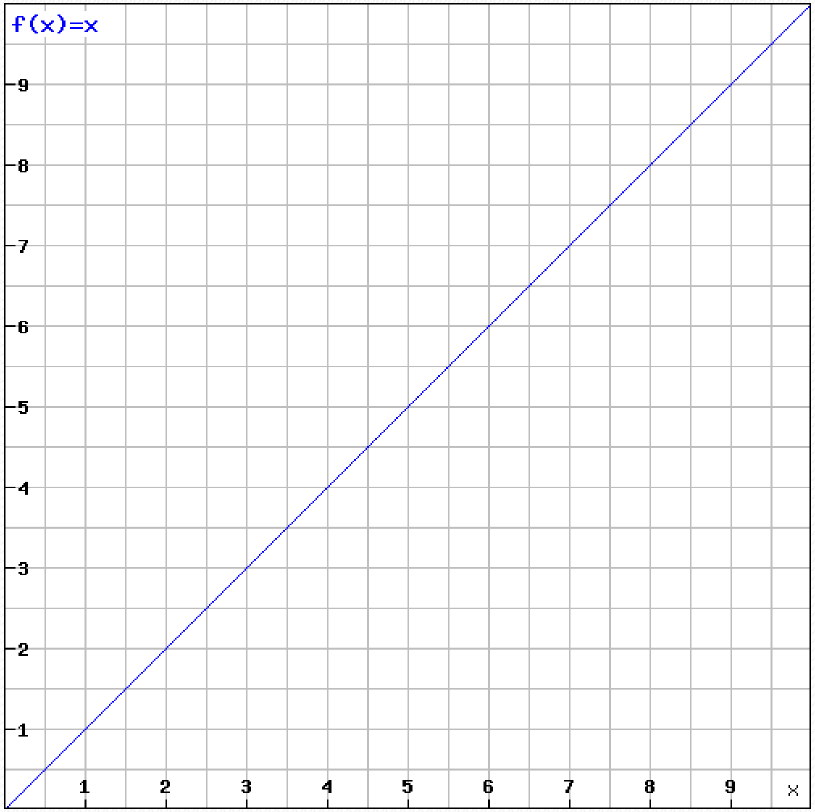
5. Calculate the following derivatives:

a) $y = 4x^2$

b) $y = 8x^3$

c) $y = 5x^2 + 2x^4$

Finding the Area under a Straight Line



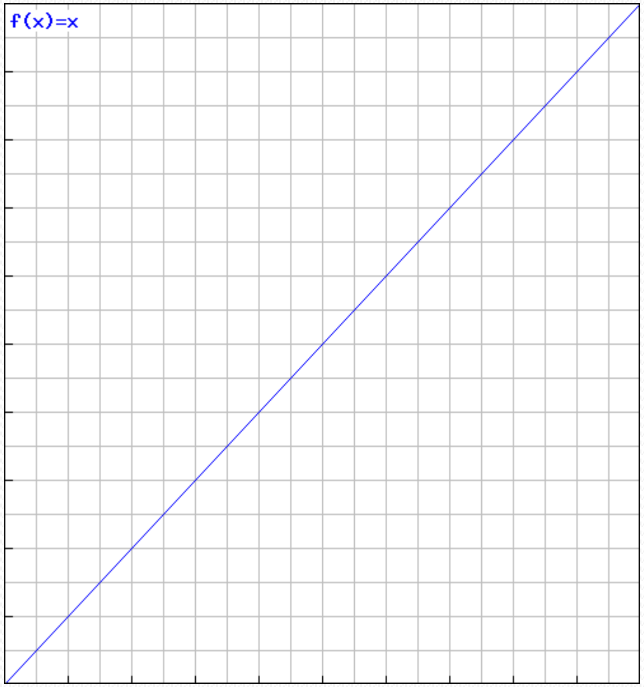
Taking two points, say x=1 and x=3 we can find the area under the line easily using the trapezium rule.

Trapezium rule: $\frac{(a+b)h}{2}$

$$\frac{(1+3)2}{2} = 4$$

Nice and simple, not very interesting

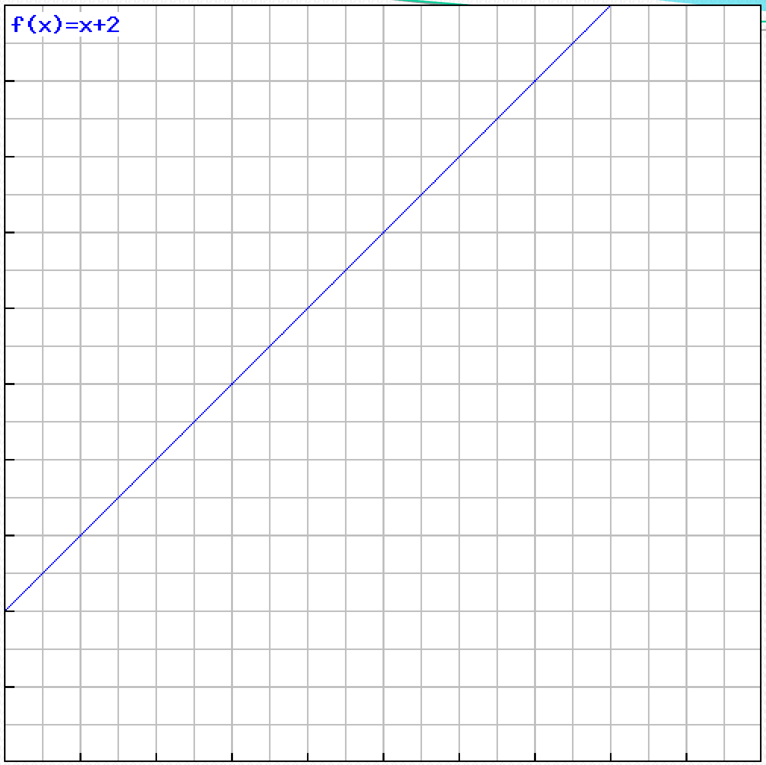
Look again at the graph y=x but with no values for axis



Again take x=1 as one of the points and take the other point just as unknown value of x

Use the same method find the area

$$Area = \frac{x^2 - 1}{2}$$



Do the same with this graph but starting point and x=2 and the other point being x

Do you notice anything about your answers?

Try differentiating them

We notice that if we differentiate the area we come back to the equation for the line

Now we think to ourselves if we can go one way surely we can go the other

Knowing the general formula for differentiation applies to any polynomial then we can assume that if there is a general formula for the reverse then that should apply to any polynomial too

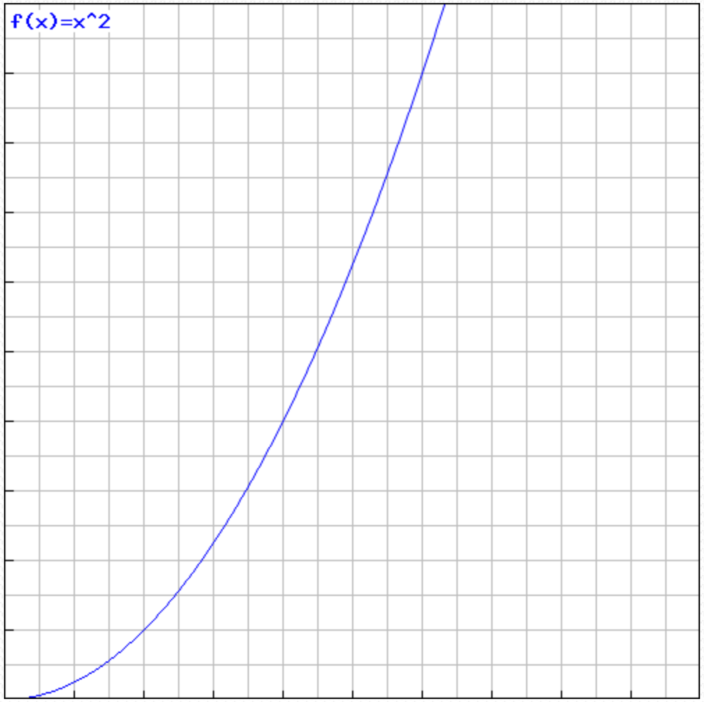
This process is called integration

Can you derive a general formula for it?

y= ax^n

A= $\frac{ax^{n+1}}{n+1}$+c

Finding the Area under a Curve



To find the are under a curve we can get an approximation using the trapezium rule over small increments

Much simpler and more accurate to use our new general formula

$y = x^2$

$$A = \frac{x^{2+1}}{2+1} = \frac{x^3}{3} + c$$

$$\begin{tikzcd} A = \frac{x^3}{3} + c \arrow[r, bend left, "\text{Differentiate}"] & y = x^2 \arrow[l, bend left, "\text{Integrate}"] \end{tikzcd}$$

Problem! We don't know what this constant c is.
We need more information

Like in the previous examples we need a range.

Going back to $y = x$ between 1 and 3 and using or rule we get:

$$A = \int_1^3 x\,dx = \left[\frac{x^2}{2}\right]_1^3 = \frac{9}{2} - \frac{1}{2} = \frac{8}{2} = 4$$

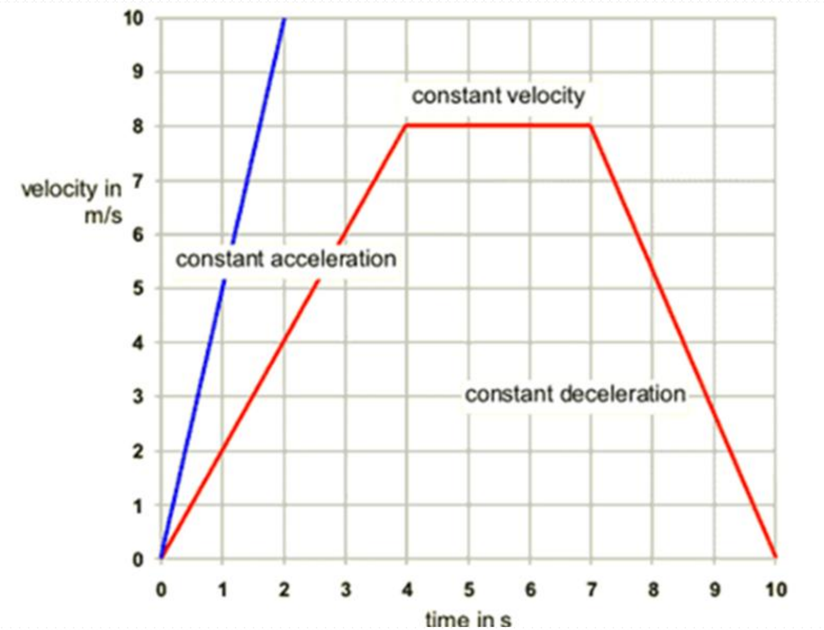
Try this for $y = x^2$ over the range between 1 and 3

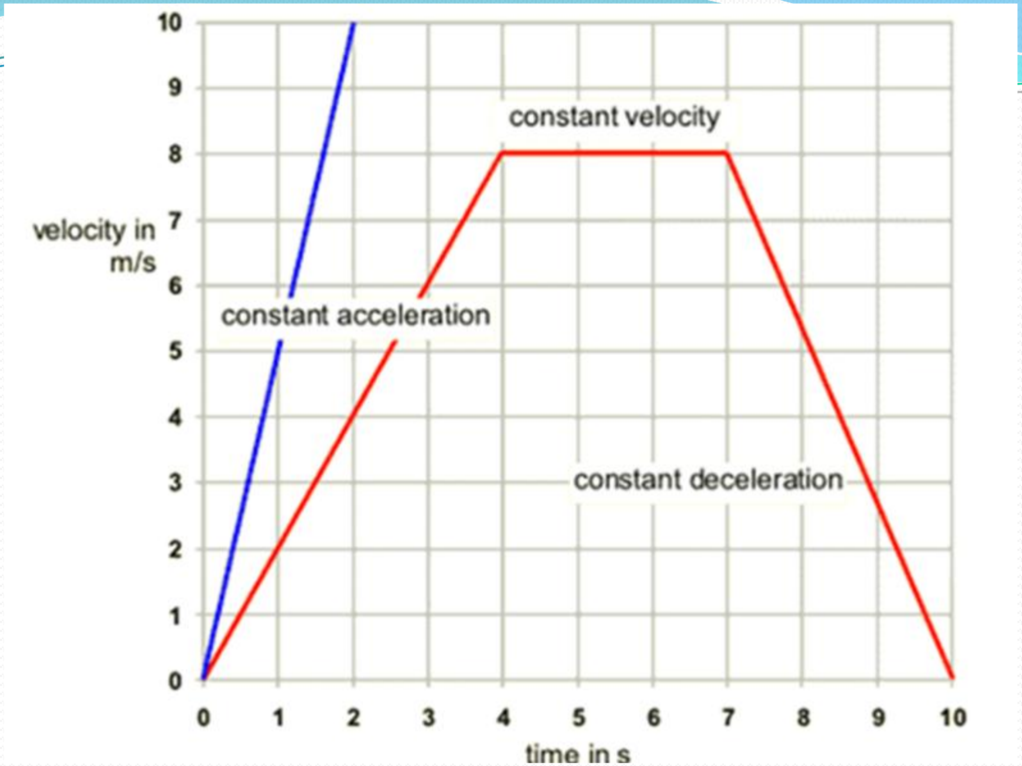
Answer should be $\frac{26}{3}$

SPEED TIME GRAPHS

Definitions:

- In scientific terms acceleration is the rate something changes its speed.
- Velocity is the speed in a particular direction
- $acceleration = \frac{change\ in\ speed}{time\ taken}$





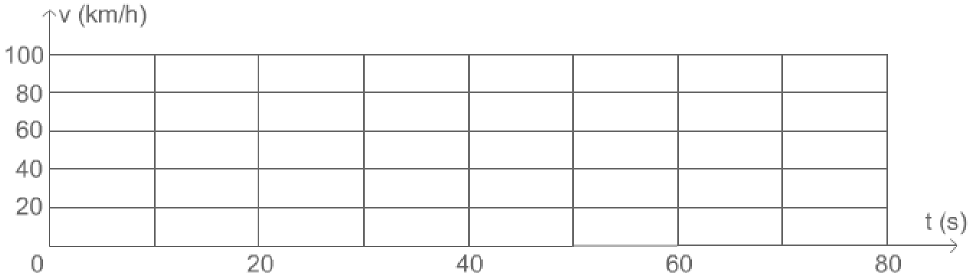
This means that the gradient of line is also the acceleration.

How does the steepness of the line affect the acceleration?

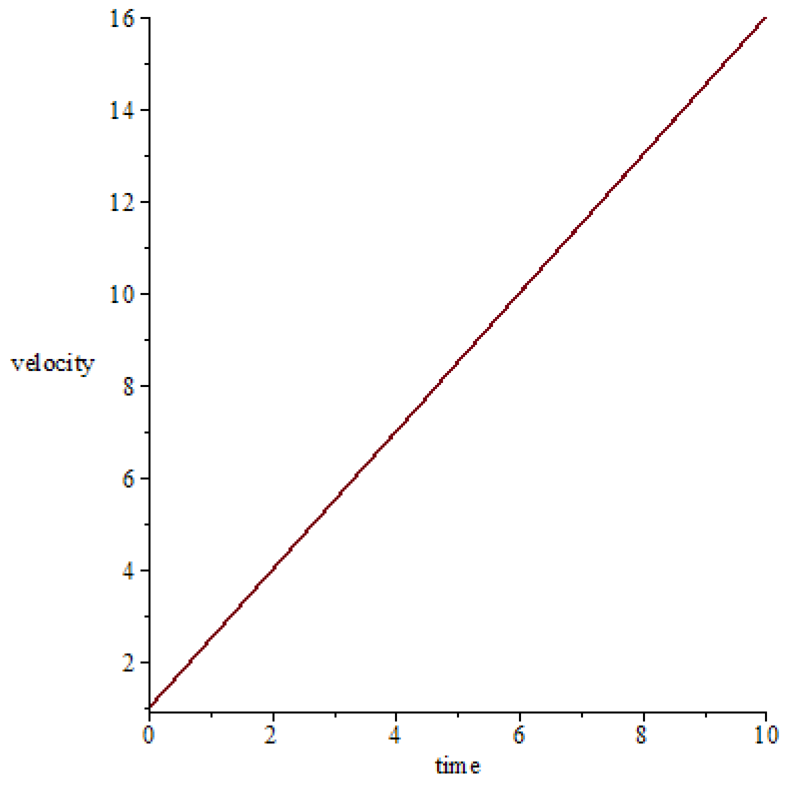
The steeper the line the greater the acceleration

What if we have a negative gradient?

We have deceleration

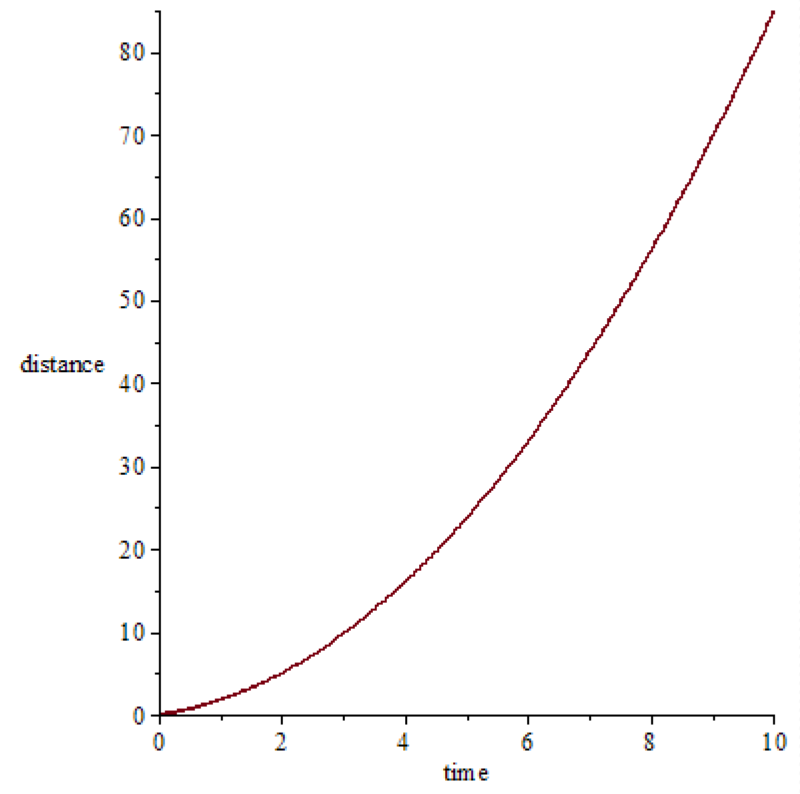


[Motorbike simulation]



The equation of this graph is $y = \frac{3x}{2} + 1$, can you integrate it over the range to find the area under the graph?

This should give us an area of 85, which is shown by the plot of distance over time graph below



This graph has equation $D = \frac{3x^2}{4} + x$